14/12/17 ST3457

(1) $Y_1, \dots, Y_6 \mid \lambda \overset{iid}{\sim} \text{Poisson}(\lambda)$

$\underline{Y} = \{16, 6, 6, 2, 4, 1\}$

i) $\lambda \sim \text{Gamma}(a, b)$

$\lambda \mid y_1, \dots, y_n \sim \text{Gamma}(a', b')$?

$$p(\lambda \mid y_1, \dots, y_n) \propto \underbrace{p(y_1, \dots, y_n \mid \lambda)}_{\text{POISSON LIK.}} \cdot \underbrace{p(\lambda)}_{\text{GAM. PRIOR}}$$

$$\dots \quad \text{Gamma}(\underbrace{a + \Sigma y_i}_{a'}, \underbrace{b + n}_{b'})$$

ii) $\lambda \sim \text{Gamma}(1, 0.2)$

BAYESIAN ESTIMATOR FOR MEAN NUMBER OF COLLISION.

$\underset{\text{POISSON}}{\mathbb{E}[Y] = \lambda}$ $\qquad \mathbb{E}[\lambda \mid y_1, \dots, y_6] =$

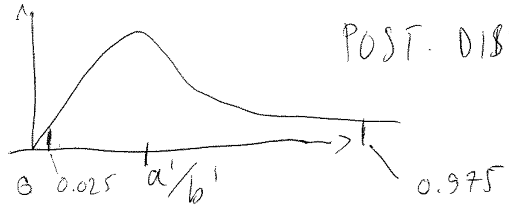
$$\Big(= \int_0^{+\infty} \lambda \cdot p(\lambda \mid y_1, \dots, y_6)\, d\lambda$$

$$= \int_0^{+\infty} \lambda \cdot \frac{(b')^{a'}}{\Gamma(a')} \cdot \lambda^{a'-1} e^{-\lambda b'} d\lambda$$

$$= \frac{(b')^{a'}}{\Gamma(a')} \int_0^{+\infty} \lambda^{(a'+1)-1} e^{-\lambda b'} d\lambda$$

$$= \frac{\cancel{(b')^{a'}}}{\Gamma(a')} \cdot \frac{\Gamma(a'+1)}{(b')^{\cancel{a'}+1}} \Big) = \frac{a'}{b'} = \frac{a + \Sigma y_i}{b + n}$$

$$= \frac{1 + 35}{0.2 + 6} = \frac{36}{6.2} = 5.806$$

iii) 

POST. DISTR. OF λ

THE GAMMA IS NOT A SYMMETRIC DISTR.
=> THE POINT ESTIMATE, IN GENERAL, WON'T BE THE MIDDLE POINT OF THE CRED. INTERVAL. (I.E. NOT SYMMETRIC AROUND THE POINT ESTIMATE)

iv) BAYESIAN ESTIMATE FOR $P[y_7 \geq 1]$

CONDITIONALLY ON λ

$$P[y_7 \geq 1] = 1 - P[y_7 = 0]$$

$$= 1 - e^{-\lambda} \cdot \frac{\lambda^0}{0!} = 1 - e^{-\lambda}$$

I.E. $P[y_7 \geq 1 \mid \lambda] = 1 - e^{-\lambda}$

BAYES ESTIMATOR: $\mathbb{E}[P[Y_7 \geq 1 \mid \lambda] \mid y_1, \dots, y_n]$

$$= \int_0^{+\infty} (1 - e^{-\lambda}) \cdot p(\lambda \mid y_1, \dots, y_n)\, d\lambda$$

$$= \int_0^{+\infty} 1 \, p(\lambda \mid y_1, \dots, y_n)\, d\lambda - \int_0^{+\infty} e^{-\lambda} p(\lambda \mid y_1, \dots, y_n)\, d\lambda$$

$$= 1 - \int_0^{+\infty} e^{-\lambda} \frac{(b')^{a'}}{\Gamma(a')} \cdot \lambda^{a'-1} e^{-\lambda b'} d\lambda$$

$$= 1 - \frac{(b')^{a'}}{\Gamma(a')} \int_0^{+\infty} e^{-\lambda(b'+1)} \cdot \lambda^{a'-1} d\lambda$$

$$= 1 - \frac{(b')^{a'}}{\cancel{\Gamma(a')}} \cdot \frac{\cancel{\Gamma(a')}}{(b'+1)^{a'}} = 1 - \left(\frac{b'}{b'+1}\right)^{a'} =$$

$$= 1 - \left(\frac{6.2}{7.2}\right)^{36} = 0.995$$

v) COND. IND = EXCHANGEABILITY

$Y_1, \dots, Y_m$ ARE EXCHANG. IF

$$P(y_1, \dots, y_m) = P(y_{\pi(1)}, \dots, y_{\pi(m)})$$ WHERE π IS A PERMUTATION

IT SEEMS THAT THE EXCHANG. ASSUMPTION IGNORES THE DECREASING TREND OF THE DATA. MAYBE NOT VERY SUITABLE FOR THESE DATA.

3) $X_1, \dots, X_8 / \lambda \overset{IID}{\sim} \mathcal{E}xp(\lambda)$

$$f(x; \lambda) = \lambda e^{-\lambda x} \mathbb{1}_{(0,+\infty)}(x)$$

$$\mathbb{E}[X] = 1/\lambda$$

i) $\lambda \sim Gamma(a, b)$

$\lambda / X_1, \dots, X_8 \sim Gamma(a', b')$?

$$p(\lambda | x_1, \dots, x_m) \propto \underbrace{p(x_1, \dots, x_m | \lambda)}_{\text{EXPONENTIAL LIKELIHOOD}} \cdot \underbrace{p(\lambda)}_{\text{GAMMA PRIOR}}$$

$$\propto \prod_{i=1}^{m} \lambda e^{-\lambda x_i} \cdot \lambda^{a-1} e^{-b\lambda}$$

$$= \lambda^m \cdot e^{-\lambda \cdot \sum x_i} \lambda^{a-1} e^{-b\lambda}$$

$$= \lambda^{a+m-1} e^{-\lambda(b + \sum x_i)}$$

$$\Rightarrow \lambda | X_1, \dots, X_m \sim Gamma(\underbrace{a+m}_{a'}, \underbrace{b + \sum x_i}_{b'})$$

ii) $X | \lambda \sim \mathcal{E}xp(\lambda)$

$$\mathbb{E}[X|\lambda] = \frac{1}{\lambda}$$

$$\mathbb{E}\left[\mathbb{E}[X|\lambda] \,|\, X_1, \dots, X_m\right] =$$

$$= \int_0^{+\infty} \frac{1}{\lambda} \cdot \frac{(b')^{a'}}{\Gamma(a')} \cdot \lambda^{a'-1} e^{-\lambda b'} d\lambda$$

$$= \frac{(b')^{a'}}{\Gamma(a')} \int_0^{+\infty} \lambda^{a'-1-1} e^{-\lambda b'} d\lambda$$

$$= \frac{(b')^{\cancel{a'}}}{\Gamma(a')} \frac{\Gamma(a'-1)}{\cancel{(b')^{a'-1}}} = \frac{b'}{a'-1}$$

iv) $p(x_9 \mid x_1, \dots, x_8) =$

$$= \int_0^{+\infty} p(x_9, \lambda \mid x_1, \dots, x_8)\, d\lambda$$

$$= \int_0^{+\infty} p(\lambda \mid x_1, \dots, x_8) \cdot p(x_9 \mid \lambda, x_1, \dots, x_8)\, d\lambda$$

$$\underset{\text{BY COND. I.I.D. ASS.}}{=} \int_0^{+\infty} p(\lambda \mid x_1, \dots, x_8) \cdot \underbrace{p(x_9 \mid \lambda)}_{\lambda e^{-\lambda x_9}}\, d\lambda$$

$$= \int_0^{+\infty} \frac{(b')^{a'}}{\Gamma(a')} \lambda^{a'-1} e^{-\lambda b'} \lambda e^{-\lambda x_9}\, d\lambda$$

$$= \frac{(b')^{a'}}{\Gamma(a')} \int_0^{+\infty} \lambda^{a'+1-1} e^{-\lambda \cdot (b + x_9)}\, d\lambda,$$

$$= \frac{(b')^{a'}}{\Gamma(a')} \cdot \frac{\Gamma(a'+1)}{(b'+x_9)^{a'+1}} = \frac{a' \cdot (b')^{a'}}{(b'+x_9)^{a'+1}} \mathbb{1}_{(0,+\infty)}(x)$$