ST2004 Applied Probability I:
sketch of the solutions of 2017 exam

1. (a) Given two events $A$ and $B$ such that $P(A) > 0$ and $P(B) > 0$,

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A \mid B)P(B) + P(A \mid B^c)P(B^c)}.$$ 

The proof is a simple application of the definition of conditional probability and the law of total probabilities.

(b) We introduce the events:

POS = the test gives positive result,
SUF = the person is a sufferer.

The information we have can be summarised as

$$P(\text{POS} \mid \text{SUF}) = \frac{95}{100}$$
$$P(\text{POS} \mid \text{SUF}^c) = \frac{1}{10}$$
$$P(\text{SUF}) = \frac{5}{1000}.$$

(i)

$$P(\text{POS}) = P(\text{POS} \mid \text{SUF})P(\text{SUF}) + P(\text{POS} \mid \text{SUF}^c)P(\text{SUF}^c)$$
$$= \frac{95}{100} \times \frac{5}{1000} + \frac{1}{10} \times \frac{995}{10000}$$
$$= \frac{10425}{100000} \approx 0.104.$$ 

(ii)

$$P(\text{SUF} \mid \text{POS}) = \frac{P(\text{POS} \mid \text{SUF})P(\text{SUF})}{P(\text{POS})}$$
$$= \frac{\frac{95}{100} \times \frac{5}{1000}}{\frac{10425}{100000}} \approx 0.046.$$
2. (a) A discrete random variable $X$ has Poisson distribution if its probability mass function is given by

$$P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!} \mathbf{1}_{\{0,1,2,\ldots\}}(k).$$

If the number of arrivals on a given interval $[0,t]$ is modelled with a Poisson distribution with parameter $\nu t$, where $\nu$ is the rate of arrivals in the unit interval, then the waiting time for the next arrival has Exponential distribution with parameter $\nu$. This is shown as follows.

Let

$$N_t = \text{number of arrivals in } [0,t]$$

$$X_t = \text{waiting time till the next arrival, starting from } t.$$

If we assume $N_t \sim \text{Poisson}(\nu t)$ then, for every $x > 0$,

$$P(X_t > x) = P(N_{t+x} - N_t = 0) = P(N_x = 0) = \frac{e^{-\nu x} \lambda^0}{0!} = e^{-\nu x}.$$

The corresponding density is given by

$$f(x) = \frac{d}{dx} (1 - P(X_t > x)) = \nu e^{-\nu x} \mathbf{1}_{(0,\infty)}(x),$$

thus showing that $X_t \sim \text{Exp}(\nu)$.

(b) We introduce the random variable

$$N_t = \text{number of phone calls in an interval of } t \text{ minutes}$$

and observe $N_t \sim \text{Poisson}(\nu t)$, with $\nu = 4$.

(i) 

$$P(N_1 > 0) = 1 - P(N_1 = 0) = 1 - \frac{e^{-\nu} \nu^0}{0!} = 1 - e^{-4} \approx 0.98.$$

(ii) 

$$P(N_1 \geq 2) = 1 - P(N_1 = 0) - P(N_1 = 1) = 1 - \frac{e^{-\frac{\nu}{2}} \left(\frac{\nu}{2}\right)^0}{0!} - \frac{e^{-\frac{\nu}{2}} \left(\frac{\nu}{2}\right)^1}{1!} \approx 0.59.$$
(iii) Let

\[ X = \text{number of minutes till the next phone call.} \]

We know that \( X \sim \text{Exp}(\nu) \) and therefore

\[ E(X) = \frac{1}{\nu} = \frac{4}{3}. \]

3. (a)

<table>
<thead>
<tr>
<th>( X = )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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</thead>
<tbody>
<tr>
<td>( Y = 1 )</td>
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<td>5/36</td>
<td>5/36</td>
<td>5/36</td>
<td>5/36</td>
<td>0</td>
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<tr>
<td>( Y = 2 )</td>
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<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>10/36</td>
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<tr>
<td>( Y = 3 )</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/36</td>
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</table>

(b) \[ P(Y = 0) = \frac{25}{36}, \quad P(Y = 1) = \frac{10}{36}, \quad P(Y = 2) = \frac{1}{36}, \]

and thus \[ E(Y) = \frac{10}{36} + \frac{2}{36} = \frac{1}{3}. \]

(c) No because, for example, \( P(X = 0, Y = 0) = 0 \neq \frac{25}{36} = P(X = 0)P(Y = 0). \)

(d) Since \( E(X) = \frac{7}{2}, \ E(Y) = \frac{1}{3} \) and \( E(XY) = \frac{57}{36} \) we conclude that

\[ \text{cov}(X,Y) = \frac{5}{12}. \]

4. The random variable

\( N_5 = \) number of defected products out of the 5 taken

can be modelled with a Binomial distribution with parameters \( n = 5 \) and \( p = 0.1. \)

(a) \[ P(N_5 = 0) = \binom{5}{0} 0.9^5 (1 - 0.9)^0 = 0.9^5 \approx 0.59. \]

(b) \[ P(N_5 \leq 1) = P(N_5 = 0) + P(N_5 = 1) \approx 0.92. \]

The random variable

\( X = \) number of defected products out of the 100 taken

can be modelled with a Binomial distribution with parameters \( n = 100 \) and \( p = 0.1. \)
(c) \[ E[X] = np = 10. \]

(d) When \( p \) is small and \( n \) is large the binomial distribution can be approximated by a Poisson distribution with parameter \( \lambda = np \), in this case \( \lambda = 10 \). Thus
\[
P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{e^{-10}10^0}{0!} + \frac{e^{-10}10^1}{1!} = 11e^{-10} \approx 4.99 \times 10^{-4}.
\]

5. (a)
\[
\text{var}(aX + b) = E((aX + b)^2) - E(aX + b)^2
= a^2E(X^2)a^2E(X)^2 = a^2\text{var}(X).
\]

(b)
\[
E(U) = \frac{1}{2},
E(U^2) = \frac{1}{3}
\]
and therefore
\[
\text{var}(U) = \frac{1}{12}.
\]

(c) We can apply the inverse transform method. Indeed if \( X \sim \text{Exp}(\lambda) \) than it can be shown that
\[
X = -\frac{1}{\lambda} \log(U).
\]
The algorithm then consists in generating \( U \) and transforming it by exploiting the previous identity so to obtain random number generation of \( X \).